

THE ADIC REALIZATION OF THE MORSE TRANSFORMATION AND THE EXTENSION OF ITS ACTION ON THE SOLENOID

A. M. VERSHIK AND B. SOLOMYAK

ABSTRACT. We consider the adic realization of the Morse transformation on the additive group of integer dyadic numbers. We discuss the arithmetic properties of that action. Then we extend that action to the action of the group of rational dyadic numbers on the solenoid.

To the memory of Alexander Livshits

Sasha Livshits (1950-2008) was the author of one of the most important theorems of modern dynamics, which is well-known now, — the theorem about cohomology of hyperbolic systems. He proved it when he was a student. Later he worked on many other problems of symbolic dynamics, ergodic theory and combinatorics. His deep and important ideas made a great impression on those who interacted with him (this includes the second author). The first author considers him the best of his students.

1. INTRODUCTION

The Morse dynamical system was discovered by Morse and popularized by Hedlund-Gottshalk. Later it was studied by many authors (see [9, 8] and references there), as a simplest nontrivial substitution. Moreover, it was historically the first example of a substitution. It is generated by the Thue-Morse sequence, which was extensively studied from the point of view of combinatorics of words and symbolic complexity (see [1, 7] and references there). The new approach to symbolic dynamics and ergodic transformations (adic transformation) which was suggested by the first author [11], can be applied also to substitutions (stationary adic transformations); this was done in the paper of A. Livshits and the first author ([13]). Later, other authors developed this connection in the context of topological dynamics, see [5, 4], but here our focus is on measure-preserving transformations. The adic realization of a substitution dynamical

Date: November 20, 2008.

system allows one to consider simultaneously not only the substitution itself but also the one-sided shift which accompanies any substitution. The idea which was advocated by the first author in [12] is to consider the natural extension of that shift and correspondingly extend the substitution system in order to make an essential link between the theory of substitutions and hyperbolic dynamics. In this paper we consider the “two-sided extension” of the Morse system which yields *the Morse action of the countable group Q_2* (the group of dyadic rational numbers) on the group of its characters — solenoid \widehat{Q}_2 , reworking more carefully and correcting some details of [12]. We obtain also some new properties of the adic realization of the Morse transformation. One of the corollaries of the adic approach is an explicit calculation which shows how to obtain the Morse system as a time-change of the dyadic odometer. The operation of differentiation of dyadic sequences plays an important role in our constructions. The spectral theory of the Morse system, which goes back to Kakutani [6] (see also [9, 8] and references there) is also becoming more transparent under these considerations, but we do not address it in this paper.

In Section 2 we collect a series of auxiliary results, which use the adic realization of the Morse system; some of them new, others are well-known, but we give their adic version. We discuss in more detail (than in [12]) the so-called “Morse arithmetic”. Section 3 describes the two-sided extension of the Morse transformation and embedding of it to the Morse action of the group Q_2 on the group \widehat{Q}_2 . We also formulate some problems.

One should consider this article as an attempt, looking at the special case of the Morse transformation, to attack the general problem of defining a two-sided extension of a substitution system, and corresponding to this extension its embedding into an action of a larger group. The final goal of the constructions is to show the link between the theory of substitutions and of hyperbolic systems.

The first author thanks to Max Plank Institute for Mathematical Physics in Bonn and Erwin Schrodinger International Institute for Mathematical Physics in Vienna where this article was finished, and Russian funds RFBR-08-01-000379a, NSh-2460.20081112. The research of the second author was supported in part by NSF grant DMS-0654408.

2. DEFINITIONS AND ADIC REALIZATIONS OF MORSE SYSTEM.

2.1. Morse as a substitution system. Consider the alphabet $\{0, 1\}$. The Morse substitution is defined by $\zeta(0) = 01$, $\zeta(1) = 10$; it is extended to words in $\{0, 1\}$ by concatenation. The Thue-Morse sequence (sometimes also called Prouhet-Thue-Morse sequence) is a fixed point of the substitution

$$(1) \quad u = u_0 u_1 u_2 \dots = \lim_{n \rightarrow \infty} \zeta^n(0) = 0110100110010110 \dots$$

It has many remarkable features, see e.g. [1] and [8, Ch.2, Ch.5]. It is easy to see that

$$u[0, 2^{n+1} - 1] = u[0, 2^n - 1] \overline{u[0, 2^n - 1]} \quad \text{for } n \geq 0,$$

where we denote $u[i, j] = u_i \dots u_j$ and \overline{w} is the “flip” of a word w in the alphabet $\{0, 1\}$, that is, we interchange $0 \leftrightarrow 1$. The sequence u is non-periodic, but uniformly recurrent, with well-defined uniform frequencies of subwords. It is also known that u_n is the sum of the digits (mod 2) in the binary representation of n .

Let σ be the left shift on the spaces $\{0, 1\}^{\mathbb{N}}$ and $\{0, 1\}^{\mathbb{Z}}$ with the product topology. The substitution dynamical system is sometimes considered on the space of one-sided sequences, and sometimes on the space of two-sided sequences.¹

The “one-sided” substitution space is defined as the orbit closure of u under the shift:

$$X_{\zeta}^+ = \text{clos}\{\sigma^n u : n \geq 0\}.$$

The “two-sided” substitution space is defined as the set X_{ζ} of all bi-infinite sequences in $\{0, 1\}^{\mathbb{Z}}$ whose every block (subword) occurs in u . The substitution dynamical systems are (X_{ζ}^+, σ) (one-sided) and (X_{ζ}, σ) (two-sided). The advantage of the two-sided system is that it is a homeomorphism, whereas the one-sided is not. Measure-theoretically these two systems are isomorphic: in fact, both are minimal and uniquely ergodic, and the one-sided system is a.e. invertible. The restriction of any point in X_{ζ} to non-negative coordinates is in X_{ζ}^+ , and all but countable many elements of X_{ζ}^+ have a unique extension to points in X_{ζ} . The exceptions are u , which extends to $u \cdot u$ and $\overline{u} \cdot u$, and its “flip” \overline{u} , as well as their orbits.

¹We have used above the notion “one-sided” and “two-sided” in completely different sense, see also below.

2.2. Adic realization. Here we follow the general definition of the adic transformation from [11] and [13], but focus only on the Morse system as it was done in [12]. Consider $\mathbf{Z}_2 \cong \{0, 1\}^{\mathbb{N}}$, the compact additive group of 2-adic integers, and the odometer (“adding machine”) transformation T , which is adic transformation by definition - this is group translation on \mathbf{Z}_2 (see below). We obtain the adic realization of Morse transformation by *changing the order of symbols 0, 1 depending on the next symbol*. Namely, consider the lexicographic order on \mathbf{Z}_2 induced by

$$0 \prec_0 1, \quad 1 \prec_1 0$$

as follows:

$$\{x_i\} \prec \{y_i\} \iff \exists j : x_i = y_i \text{ for } i > j \text{ and } x_j \prec_z y_j, \text{ where } z = x_{j+1} = y_{j+1}.$$

This is a partial order; two sequences are comparable if they are cofinal (agree except in finitely many places). The set of maximal points is $\mathbf{Max} = \{(01)^\infty, (10)^\infty\}$, and the set of minimal points is $\mathbf{Min} = \{(0)^\infty, (1)^\infty\}$.² Let M be the immediate successor transformation in the order \prec on \mathbf{Z}_2 . Here we write down the formulas for the action of M explicitly. If $x \notin \mathbf{Max}$, then x starts with $(01)^n 00$ or $(01)^n 1$, or $(10)^n 0$, or $(10)^n 11$, where $n \geq 0$. We have

$$(2) \quad \begin{aligned} M((01)^n 00*) &= (1^{2n+1} 0*), & M((01)^n 1*) &= (0^{2n} 1*), \\ M((10)^n 0*) &= (1^{2n} 0*), & M((10)^n 11*) &= (0^{2n+1} 1*) \end{aligned}$$

Note that it is well-defined everywhere except on the two maximal points: $\mathbf{Max} = \{(01)^\infty, (10)^\infty\}$. It is easy to see that M is continuous on $\mathbf{Z}_2 \setminus \mathbf{Max}$. But it is impossible to extend M by continuity to those points: there are no well-defined limits of $\lim_{n \rightarrow \infty} M((01)^n *)$, and $\lim_{n \rightarrow \infty} M((10)^n *)$, because

$$\lim_{n \rightarrow \infty} M((01)^n 00*) = (1)^\infty,$$

but

$$\lim_{n \rightarrow \infty} M((01)^n 1*) = (0)^\infty.$$

Analogously,

$$\lim_{n \rightarrow \infty} M((10)^n 00*) = (1)^\infty,$$

²We denote infinite periodic sequence with period $(ab \dots c)$ as $(ab \dots c)^\infty$.

but

$$\lim_{n \rightarrow \infty} M((10)^n 1*) = (0)^\infty.$$

Since we also have two minimal points, we can extend M to a bijection arbitrarily, setting

$$(3) \quad M((01)^\infty) = (1)^\infty, \quad M((10)^\infty) = (0)^\infty,$$

or vice versa. But this extension is not continuous at those points.

The obvious corollary of the definition of M is that it commutes with “flips,” that is,

$$(4) \quad M(\overline{x}) = \overline{M(x)} \quad \text{for all } x \in \mathbf{Z}_2.$$

The action of M on $\mathbf{Z}_2 \setminus \mathbf{Max}$ may be expressed as follows: we scan the sequence x to the right until we see two identical symbols aa and replace the beginning of the sequence by $\overline{a} \dots \overline{a}a$, keeping the second occurrence of a and everything that follows unchanged.

2.3. The relation of adic model to traditional representation. Now we indicate the relation between the dynamical systems (\mathbf{Z}_2, M) and (X_ζ, σ) . Let

$$g : \mathbf{Z}_2 \rightarrow X_\zeta, \quad g(x) = \{(M^{n-1}x)_0\}_{n \in \mathbb{Z}}.$$

We have the following diagram:

$$\begin{array}{ccc} \mathbf{Z}_2 & \xrightarrow{M} & \mathbf{Z}_2 \\ \downarrow g & & \downarrow g \\ X_\zeta & \xrightarrow{\sigma} & X_\zeta \end{array}$$

It is obvious from the definition that the diagram commutes. It is also easy to see that g is surjective, continuous on $\mathbf{Z}_2 \setminus \widetilde{\mathbf{Max}}$, and $g(0^\infty) = \overline{u} \cdot u$. Here we denoted by $\widetilde{\mathbf{Max}}$ the set of points in \mathbf{Z}_2 which are cofinal with the points in \mathbf{Max} (equivalently, their left semi-orbit).

It may be useful to write down g^{-1} explicitly. Consider the substitution map on X_ζ :

$$\zeta : X_\zeta \rightarrow X_\zeta, \quad \zeta(\dots a_{-2}a_{-1} \cdot a_0a_1 \dots) = \dots \zeta(a_{-2})\zeta(a_{-1}) \cdot \zeta(a_0)\zeta(a_1) \dots$$

It is well-known (and easy to see) that for every $\mathbf{a} \in X_\zeta$ there is a unique $\mathbf{a}' \in X_\zeta$ such that either $\mathbf{a} = \zeta(\mathbf{a}')$ or $\mathbf{a} = \sigma\zeta(\mathbf{a}')$, and these cases are mutually exclusive. Let

$\Psi : X_\zeta \rightarrow X_\zeta$ be given by $\Psi(\mathbf{a}) = \mathbf{a}'$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Z}_2 & \xrightarrow{\sigma} & \mathbf{Z}_2 \\ \downarrow g & & \downarrow g \\ X_\zeta & \xrightarrow{\Psi} & X_\zeta \end{array}$$

where σ is the left shift on \mathbf{Z}_2 . Therefore, to compute the n -th symbol of $g^{-1}(\mathbf{a})$ we need to take $(\Psi^n(\mathbf{a}))_0$ for $n = 0, 1, 2, \dots$

Now let us explain why this model is richer than the “two-sided” model X_ζ . In the adic realization, we have the adic transformation, which is isomorphic, up to neglecting of two orbits, to the substitution, AND we have the one-sided shift in the space \mathbf{Z}_2 . Evolution under the adic transformation of the first digit x_0 of the sequence $\{x_i\} \in \mathbf{Z}_2$ just gives the orbit of u under the transformation σ on X_ζ . The one-sided shift in the space \mathbf{Z}_2 in terms of substitutions is the substitution itself — i.e. the transformation of any sequence which substitutes 0 by 01 and 1 by 10. So we have in the adic model a simultaneous realization of both transformations: the adic and the shift (substitution). The problem arises how to include in this picture the natural extension of the one-sided shift — the two-sided shift, and at the same time how to extend the adic transformation to the whole space. We will do this in the next section but first we interpret a familiar property of the Morse system in our terms.

2.4. Morse as a 2-point extension of the odometer.

Definition 2.1. *The classical 2-odometer is the following affine transformation T on the additive group \mathbf{Z}_2 of dyadic integers:*

$$T : Tx = x + 1.$$

The transformation T preserves the Haar (=Bernoulli, Lebesgue) measure on the group \mathbf{Z}_2 . It is well-known that the Morse system can be represented as a group (2-point) extension of the dyadic odometer. This is the most popular point of view on the Morse system in dynamics. The adic realization of the Morse transformation gives another way to look at this fact; it is an illustration of the importance of the so-called differentiation of sequences.

Define the important map:

Definition 2.2. *Differentiation of sequences is the map $D : \mathbf{Z}_2 \rightarrow \mathbf{Z}_2$:*

$$D(\{x_n\}_{n=0}^\infty) = \{(x_{n+1} - x_n) \bmod 2, n = 0, 1, \dots\}.$$

This is nothing other than a 2-to-1 factorization of \mathbf{Z}_2 on itself. It is clear that the differentiation commutes with the “flip” which has been defined above: $D(\bar{x}) = D(x)$.

In spite of simplicity of the definition of the map $D : \mathbf{Z}_2 \rightarrow \mathbf{Z}_2$, there are no good and simple “arithmetic” or “analytic” expressions for the description of $D(\cdot)$. Recently V. Arnold for different reasons made many experiments on the behavior of 0-1 sequences under the iteration of differentiation [2]. But the most important thing for us is that the map D takes the Morse transformation into the odometer:

Proposition 2.3. *The following equality takes place: $T \circ D = D \circ M$.*

This is an immediate corollary of (2). So in the adic realization, the Morse transformation M is a 2-covering of the odometer in its algebraic form. Let us give a precise description of the equivalence between the Morse transformation and a 2-extension of the odometer. Let $F(x) = (Dx, x_0)$ be the map from \mathbf{Z}_2 to $\mathbf{Z}_2 \times \{0, 1\}$. This is a bijection, and we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Z}_2 & \xrightarrow{M} & \mathbf{Z}_2 \\ \downarrow F & & \downarrow F \\ \mathbf{Z}_2 \times \{0, 1\} & \xrightarrow{T(\phi)} & \mathbf{Z}_2 \times \{0, 1\} \end{array}$$

Here $T(\phi)$ is a 2-extension of T with a cocycle ϕ on Z_2 defined by

$$(5) \quad \phi(y) = \begin{cases} 0 & \text{if } y \text{ starts with odd number of 1's} \\ 1 & \text{if } y \text{ starts with even number of 1's} \end{cases}$$

To make it work on maximal elements we also need to set $\phi(1^\infty) = \phi(0^\infty) = 1$. Recall that the group extension is defined by

$$T(\phi)(x, g) = (Tx, \phi(x) + g).$$

We have

$$M = F^{-1}T(\phi)F,$$

so M is canonically isomorphic to a 2-extension of the odometer T with cocycle ϕ . We can identify \mathbf{Z}_2 with $\mathbf{Z}_2 \times \{0, 1\}$ regarding the second component $\{0, 1\}$ as a new digit

of a sequence; then the map F becomes a transformation of \mathbf{Z}_2 , and we can consider the extension inside the group \mathbf{Z}_2 . We give another interpretation of this cocycle in the next section.

Remark. The Morse system can also be realized as a 2-point extension of the odometer in the traditional substitution form, and it is interesting that the projection is again given by the differentiation map. This follows from the fact that for the Thue-Morse sequence u (see (1)), its derivative sequence $D(u) = 1011101010\dots$ is the fixed point of the substitution $0 \rightarrow 11$, $1 \rightarrow 10$ (see [1, p.201]), which generates a measure-preserving transformation isomorphic to the 2-odometer.

Denote by S the map $x \rightarrow 2x$ on \mathbf{Z}_2 , this is nothing other than the one-sided non-invertible shift or Bernoulli endomorphism if we represent the elements of \mathbf{Z}_2 as a sequence of 0's and 1's. It is easy to check the following fact.

Proposition 2.4. *The 2-odometer as well as the Morse transformation satisfy the following equation:*

$$TS = ST^2, \quad MS = SM^2.$$

2.5. Morse as a time change of the odometer and Morse arithmetic. Because the group of rational integers \mathbb{Z} is a dense invariant subgroup of the group of dyadic integers, we can consider the Morse transformation M in the adic realization as a map of integers to itself. This subsection is based on [12, p.538], but we provide more details.

Let us identify a sequence $x_0x_1x_2\dots$ with the dyadic decomposition of a number: $\sum_j x_j 2^j$. Here is the list of several first values of $M(n)$:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
1	3	7	2	5	15	4	6	9	11	31	10	13	8	12	14	...

The table can be easily verified using (2).

To this end, we introduce the following sequence:

$$a_r = \begin{cases} \frac{2^r-1}{3} & \text{if } r \equiv 0 \pmod{2}, \\ \frac{2^r-2}{3} & \text{if } r \equiv 1 \pmod{2}. \end{cases}$$

Each $n \in \mathbb{N}$ is uniquely represented in one of the following ways ($r = r(n)$):

$$(6) \quad n = \begin{cases} 2^r \ell + a_{r-1} & \text{(i),} \\ 2^r \ell + 2^{r-1} + a_r & \text{(ii),} \end{cases}$$

where $\ell \geq 0$ is an integer. Define the mapping $M : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$ by

$$(7) \quad M(n) = \begin{cases} n + a_{r(n)} & \text{in the case (i),} \\ n - a_{r(n)} & \text{in the case (ii).} \end{cases}$$

Although these formulas look a bit mysterious, they easily follow from (2). In fact,

$$a_r = \frac{2^r - 1}{3} = (10)^{(r-2)/2}, \quad r \equiv 0 \pmod{2}, \quad a_r = \frac{2^r - 2}{3} = (01)^{(r-1)/2}, \quad r \equiv 1 \pmod{2}.$$

Case (i) above occurs when the first pair aa in the binary representation of n is 00. Then $M(n)$ replaces the beginning of the sequence with 1's, which increases the number by $a_{r(n)}$ (observe that $a_{r-1} + a_r = 2^{r-1} - 1 = (1)^{r-1}$ independent of the parity of r). Similarly, the case (ii) above occurs when the first pair aa in the binary representation of n is 11. In this case $M(n)$ decreases or increases the number n by $a_{r(n)}$.

Thus, we have described independently the restriction of the adic Morse system to \mathbb{N} ; that is why we use the same symbol M . Define M for negative integers by $M(-n) = -M(n-1) - 1$. Then it is easy to check that we have the properties $M(\overline{x}) = \overline{M(x)}$ where $\overline{n} = -n - 1$; this should be understood by identifying integers with their binary expansions. Thus, we have $M : \mathbb{Z} \rightarrow \mathbb{Z} \setminus \{0, -1\}$. Note that $0 = (0)^\infty$ and $-1 = (1)^\infty$ are the two minimal points in our ordering on \mathbf{Z}_2 . According to (3),

$$M(-1/3) \equiv M((10)^\infty) = (0)^\infty \equiv 0 \quad \text{and} \quad M(-2/3) \equiv M((01)^\infty) = (1)^\infty \equiv -1.$$

2.6. The orbit equivalence of Morse system and 2-odometer. The orbit of the point $x \in X$ with respect to an invertible transformation S of X is a set of $S^n x$, $n \in \mathbb{Z}$. Evidently, the T -orbit of any point $x \in \mathbf{Z}_2$ which has infinitely many 0's and 1's is the set of all points which are eventually equal to x . The set of all points which have finitely many 0's or 1's make one orbit (this is the common T -orbit of $(0)^\infty$ and $(1)^\infty$). Let us describe the orbit partition of Morse transformation, which follows directly from the definition (2).

Proposition 2.5. *If a point $x \in \mathbf{Z}_2$ has infinitely many subwords 00 and subwords 11, then the M -orbit of x is the set of all points which are eventually equal to x ;*

the remaining countable set of the points which have finitely many subwords 00 or subwords 11, is just the union of four semiorbits of M : two positive M -semiorbits (\mathbb{Z}_+ M -orbit) — of the point $(0)^\infty$ and point $(1)^\infty$, and two negative M -semiorbits (\mathbb{Z}_- M -orbit) — of the point $(10)^\infty$ and point $(01)^\infty$.

Note that the negative M -semiorbit of $(10)^\infty$ (correspondingly $(01)^\infty$) consists of the set of points which are eventually $(10)^\infty$ (correspondingly $(01)^\infty$) and have an initial even word.

Corollary 2.6. *Orbit partitions of the 2-odometer and the adic realization of the Morse transformation coincide (mod 0) with respect to the Haar (Lebesgue) measure on \mathbf{Z}_2 .*

As we saw, they coincide on the complement of a countable set. We will refine this claim below.

Using our extension of M by the definition (3) we can make an additional remark about those four semiorbits; we do not use it later. Note that two positive M -semiorbits generate one T -orbit, and each negative M -semiorbit is a full T -orbit. So in our definition (3) we cut one T -orbit of $(0)^\infty$ and $(1)^\infty$ and glue the first part $(0)^\infty$ with the M -semiorbit of the point $(10)^\infty$, and the second part $(1)^\infty$ with the M -semiorbit of the point $(01)^\infty$.

If $x \in \mathbb{N} \subset \mathbf{Z}_2$ then we can write tautologically

$$M(n) = T^{M(n)-n}(n),$$

where in the left-hand side $M(n)$ is the image of n under the transformation M , and in the right-hand side $M(n)$ is a natural number. Now observe that by the definition of the action of the Morse automorphism M on the set of integers defined by (7) above we have:

$$M(n) - n = (-1)^{r(n)} \cdot a_{r(n)}.$$

It is worth mentioning here that the value of the cocycle $\phi(n)$ from the previous subsection is exactly $M(n) - n \pmod{2}$, e.g. it is equal to 0 iff n and $M(n)$ have the same parity.

Denote

$$\theta(n) = (-1)^{r(n)} \cdot a_{r(n)},$$

then we have the formula

$$M(n) = T^{\theta(n)}n$$

for each rational integer n . It is clear that the function $r(\cdot)$ and consequently the function $\theta(\cdot)$ can be extended from positive integers \mathbb{N} to the group of all dyadic integers \mathbf{Z}_2 as follows: formula (5) makes sense for all $x \in \mathbf{Z}_2$ with some $r \in \mathbb{N}$ and $\ell \in \mathbf{Z}_2$, not only for integers x , with the same definition. We just consider infinite sequences of x_n . So $\theta(\cdot)$ becomes a function on \mathbf{Z}_2 with integer values; we can say that this is simply the extension of $\theta(\cdot)$ by continuity in the pro-2-topology.

We have proved the following:

Theorem 2.7. *Let M be the adic realization of the Morse transformation in the space \mathbf{Z}_2 . Let $\widetilde{\text{Max}} \cup \widetilde{\text{Min}}$ be the countable set which is union of the semi-orbits under the action of M of the four points of \mathbf{Z}_2 :*

$$(0)^\infty, (1)^\infty, (01)^\infty, (10)^\infty.$$

Then on the M -invariant set $\mathbf{Z}_2 \setminus (\widetilde{\text{Max}} \cup \widetilde{\text{Min}})$ the odometer $T : Tx = x + 1$ and the Morse transformation M have the same orbit partition, and moreover,

$$Mx = T^{\theta(x)}x \quad \text{for } x \in \mathbf{Z}_2 \setminus (\widetilde{\text{Max}} \cup \widetilde{\text{Min}}),$$

where $\theta(x)$ is the function defined above.

The formula above gives an independent definition of the Morse transformation using a time change of the odometer.

Dye's Theorem asserts that any ergodic automorphism S is isomorphic (mod 0) to an automorphism which is a time change of the odometer T (or any other given ergodic automorphism): $Sx = T^{\theta(x)}(x)$. Nevertheless, there are few examples of an explicit formula for such a time change function $\theta(\cdot)$. The theorem above is just of this type: the Morse automorphism is represented as a time change of the dyadic odometer. It is also known (see [3], theorem 3.8) that the time change integer-valued function $\theta(\cdot)$ cannot be integrable if ergodic automorphisms have the same orbits, unless $T = S$ or $T = S^{-1}$. It is easy to check that our function θ is indeed non-integrable because it has exactly two singularities on the space \mathbf{Z}_2 at the points $(01)^\infty (\equiv -1/3)$, and $(10)^\infty (\equiv -2/3)$, and the measure of the cylinder on which the values of $\theta(x)$ are equal to a_r is of order $C2^{-r}$, hence the singularities have the type of simple poles $1/t$. The weakness (closeness

to integrability) of these singularities shows that the Morse automorphism is in a sense very close to the odometer, i.e. to an automorphism with discrete spectrum.

Question. What is the group generated by two transformations of \mathbf{Z}_2 — the odometer T and the Morse transformation M ? Is it a free group?

3. EXTENSION OF THE MORSE TRANSFORMATION UP TO ACTION OF THE GROUP Q_2 ON THE SOLENOID

In this section we define the so-called two-sided extension of the Morse transformation which acts on the group of characters of dyadic rational numbers. It is an elaboration of [12, p.539] with important changes and additions.

3.1. Preliminary facts about dyadic groups Q_2 , \widehat{Q}_2 , etc. Consider the exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow Q_2 \longrightarrow Q_2/\mathbb{Z} \longrightarrow 1$$

where Q_2 is the countable additive group of real dyadic rational numbers ($r/2^m, r \in \mathbb{Z}, m \geq 0$), the subgroup $\mathbb{Z} \subset Q_2$ is the group of rational integers, and the quotient group Q_2/\mathbb{Z} is the group of all the roots of unity of orders 2^n , $n = 0, 1, \dots$ (a subgroup of the rotation group).

The group Q_2 can be presented as an inductive limit

$$\varinjlim (\mathbb{Z}, w_n),$$

of the groups \mathbb{Z} , with the embedding of n -th group given by

$$w_n(x) = 2x, \quad n = 0, 1, \dots$$

Consider the corresponding dual exact sequence for the groups of characters of the groups above:

$$1 \longleftarrow \mathbb{R}/\mathbb{Z} \longleftarrow \widehat{Q}_2 \longleftarrow \mathbf{Z}_2 \longleftarrow 1$$

The group of characters of the group Q_2/\mathbb{Z} is just the additive group of dyadic integers, \mathbf{Z}_2 , which we considered in the previous sections, and which is the inverse limit of 2^n -cyclic groups:

$$\mathbf{Z}_2 = \varprojlim (\mathbb{Z}/2^n, p_n),$$

with the maps $p_n : \mathbb{Z}/2^n \rightarrow \mathbb{Z}/2^{n-1}$, $p_n(x) = x \bmod 2^{n-1}$. The group of characters of the group \mathbb{Z} is the rotation group $S^1 = \mathbb{R}/\mathbb{Z}$ (or the unit circle).

Our main object — the group \widehat{Q}_2 of characters of the group Q_2 , is the so-called *2-solenoid* and can be presented as an inverse limit of the rotation groups:

$$\widehat{Q}_2 = \varprojlim_n (\mathbb{R}/\mathbb{Z}, v_{n+1}), \quad n = 0, 1, \dots$$

where the homomorphisms are

$$v_n : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}, \quad v_n(u) = 2u, \quad n = 1, 2, \dots$$

The group \mathbf{Z}_2 is a closed subgroup of the group \widehat{Q}_2 of those elements which have the zero projection on \mathbb{R}/\mathbb{Z} equal to 0.

The additive group Q_2 of dyadic rational numbers is naturally embedded into \widehat{Q}_2 as a dense subgroup; it consists of those characters which send the elements of the group Q_2 to the roots of unity of degree 2^n .

Note that the additive group of the locally compact field \mathbf{Q}_2 of all 2-adic number is naturally embedded into the solenoid \widehat{Q}_2 as a dense subgroup of those elements of \widehat{Q}_2 , which have the projection under the map $\widehat{Q}_2 \rightarrow \mathbb{R}/\mathbb{Z}$ to a root of unity of degree 2^n for some n :

$$\mathbf{Q}_2 \subset \widehat{Q}_2.$$

As a compact group, \widehat{Q}_2 has a normalized Haar measure which is the product of Haar measures on the groups \mathbf{Z}_2 and \mathbb{R}/\mathbb{Z} .

The group \widehat{Q}_2 is not the direct product of the subgroups \mathbf{Z}_2 and the quotient group \mathbb{R}/\mathbb{Z} ; there is a nontrivial 2-cocycle \mathbf{Z}_2 of the group \mathbb{R}/\mathbb{Z} with values in \mathbb{Z} (integer part of product). But there is a non-algebraic decomposition into the direct product of the subgroup and the quotient group:

$$\widehat{Q}_2 = \mathbb{R}/\mathbb{Z} \times \mathbf{Z}_2.$$

Here the first component of the decomposition is realized as the first unit circle (for $n = 0$ in the definition of the projective limit above). The second component is the subgroup $\mathbf{Z}_2 \subset \widehat{Q}_2$ which consists of the elements which have the second coordinate in the decomposition equal to $1 \in \mathbb{R}/\mathbb{Z}$. This decomposition gives the coordinatization of the group \widehat{Q}_2 .

Because the group Q_2 of dyadic rational numbers can be represented as the group of all finite on both sides two-sided sequences of 0's and 1's with the usual binary expansion over $2^n, n \in \mathbb{Z}$, one can think that the analog of such decomposition for the group \widehat{Q}_2 is also true. Moreover, we have used one-sided sequences with positive indices for parametrizations of the elements of the subgroup \mathbf{Z}_2 , and that parametrization agrees with the group structure of 2-adic integers. Thus, it is tempting to consider the whole group \widehat{Q}_2 of characters of the group Q_2 as a compact space of all two-sided infinite $\{0, 1\}$ sequences: $\mathbf{X} = \prod_{-\infty}^{+\infty} \{0, 1\} = \{0, 1\}^{\mathbb{Z}}$. But this is not correct, because there is no needed group structure on the space \mathbf{X} . Nevertheless, it is possible to define a map $\pi : \mathbf{X} \rightarrow \widehat{Q}_2$ with the help of the usual dyadic decomposition of the points of the unit interval $(0, 1)$ as follows: let $\{x_n\}, n \in \mathbb{Z}$, be a point of \mathbf{X} ; define the pair (y, λ) , where $y \in \mathbf{Z}_2$ is generated by the sequence $\{x_n, n \geq 0\}$ as in Section 1,

$$\lambda = \sum_{n=1}^{\infty} x_{-n} 2^{-n}.$$

Denote this map by π :

$$(8) \quad \pi : \prod_{-\infty}^{+\infty} \{0, 1\} \longrightarrow \widehat{Q}_2, \quad \pi : \{x_n\} \mapsto (y, \lambda).$$

The map π is not an isomorphism of the groups or even topological spaces but trivially *is* an isomorphism (mod 0) of the measure spaces, where the measure on the space \mathbf{X} is the $(1/2, 1/2)$ Bernoulli (product) measure, and on the group \widehat{Q}_2 , it is the Haar measure. So, if we ignore the group structure of \widehat{Q}_2 and consider it not as solenoid but as a symbolic space with measure-preserving transformations (odometer, Morse, etc.), then it is convenient to use the canonical map $\pi; \prod_{-\infty}^{+\infty} \{0, 1\} \rightarrow \widehat{Q}_2$, which identifies only countably many pairs of points. Roughly speaking, we can consider the 2-solenoid \widehat{Q}_2 as the space \mathbf{X} of all two-sided sequences 0's and 1's after some identifications of elements from the negative (left) side, which (identification) corresponds to the non-uniqueness of dyadic decomposition on the left side.

3.2. Some transformations and differentiation on the solenoid. There is a canonical automorphism \widehat{S} on the group \widehat{Q}_2 : the multiplication by 2; it is conjugate to the automorphism S^* of the group Q_2 — the multiplication by $1/2$. The transformation \widehat{S} is a hyperbolic automorphism of the solenoid and in the usual coordinatization it is

just the Bernoulli 2-shift and *natural extension* in Rokhlin sense [10] of the one-sided shift S of the space \mathbf{Z}_2 , which was defined in the section 2.

Now we define the two-sided version of 2-odometer. Let 1 be the unit of the ring \mathbf{Z}_2 (unity of the multiplicative group). We extend the odometer T from Section 2, using notation \widehat{T} , by the same formula

$$\widehat{T}x = x + 1,$$

where x now is an element of \widehat{Q}_2 . It is useful to keep in mind that 1 is a character of the group Q_2 which sends integers $\mathbb{Z} \subset Q_2$ to 1.

The action of \widehat{T} does not change the second (left) component in the decomposition $\widehat{Q}_2 = Z_2 \times \mathbb{R}/\mathbb{Z}$, so it is indeed an extension of T . Note that \widehat{T} is not an ergodic transformation of \widehat{Q}_2 , whereas T is ergodic on \mathbf{Z}_2 . We can define also the family of odometer-transformations

$$T_0 := \widehat{T}, \quad T_i := S^i T_0 S^{-i}, \quad i \in \mathbb{Z}.$$

It is clear that T_i and T_j commute and its joint action on \widehat{Q}_2 is the action of the group Q_2 on \widehat{Q}_2 ; this is the translation on the corresponding elements, as mentioned above. We claim that

$$(9) \quad T_i^2 = T_{i+1}, \quad i \in \mathbb{Z}.$$

Indeed, this is immediate for $i = 0$ and hence for all i .

Together with the shift \widehat{S} the odometers T_i generate a solvable group (wreath product) $\mathbb{Z} \ltimes \sum_{\mathbb{Z}} \mathbb{Z}$; the action of this group on the group \widehat{Q}_2 is continuous and local-transversal in the sense of the paper [12].

Define the *differentiation* \widehat{D} as a transformation of \mathbf{X} which extends the map D to the space of two-sided sequences.

$$\widehat{D}(\{x_n\}_{-\infty}^{+\infty}) = \{(x_n - x_{n+1}) \pmod{2}\}$$

We can, of course, define the differentiation on the solenoid \widehat{Q}_2 by $\widetilde{D} = \pi \circ \widehat{D} \circ \pi^{-1}$, which is well-defined almost everywhere. Observe that \widehat{D} identifies a two-sided sequence \tilde{x} with its “flip” $\bar{\tilde{x}}$, and hence almost everywhere on \widehat{Q}_2 we have

$$\widetilde{D}(y, \lambda) = \widetilde{D}(z, \gamma) \iff (y, \lambda) = (z, \gamma) \text{ or } (y, \lambda) = (\bar{z}, 1 - \gamma).$$

As we mentioned above, it is difficult to give a precise formula for \widetilde{D} in terms of characters.

3.3. Extension of the Morse transformation. Now we would like to extend the Morse transformation M from the subgroup \mathbf{Z}_2 to the whole group \widehat{Q}_2 and space \mathbf{X} .

We want to have the following properties of \widehat{M} : it must be a 2-extension of the extended odometer \widehat{T} , namely, the relation generalizing Proposition 2.1 must be valid:

$$(10) \quad \widehat{T} \circ \widehat{D} = \widehat{D} \circ \widehat{M},$$

and it should be an extension:

$$\widehat{M}|_{\mathbf{Z}_2} = M.$$

Theorem 3.1. *There is a unique transformation of the space \mathbf{X} which satisfies the last two equations. Then it defines a measure-preserving transformation \widetilde{M} on \widehat{Q}_2 via $\widetilde{M} = \pi \circ \widehat{M} \circ \pi^{-1}$, where π is defined by (8).*

Proof. The uniqueness is clear, and the existence can be shown as follows. The sequences of the space \mathbf{X} can be divided into positive and negative parts: for $\hat{x} = (\dots x_{-1}, x_0, x_1 \dots)$ denote $x_- = (\dots x_{-2}, x_{-1})$ and $x_+ = (x_0, x_1 \dots)$. Then we can define

$$(11) \quad \widehat{M}(\hat{x}) \equiv \widehat{M}((x_-, x_+)) = \begin{cases} (x_-, M(x_+)), & \text{if } \phi(x_+) = 0 \\ (\overline{x_-}, M(x_+)), & \text{if } \phi(x_+) = 1. \end{cases}$$

Here ϕ is the cocycle defined in (5). Verification of (10) is immediate.

On the solenoid \widehat{Q}_2 we get an explicit formula for the Morse transformation:

$$\widetilde{M}(y, \lambda) = (My, \lambda) \text{ if } \phi(y) = 0, \quad \widetilde{M}(y, \lambda) = (Ma, 1 - \lambda) \text{ if } \phi(y) = -1.$$

(Here y corresponds to x_+ in \mathbf{X} .) □

Observe that \widehat{M} is not “local” in the sense that it does change negative coordinates when the cocycle does not vanish.

The extension \widehat{M} is continuous on $\{x \in \widehat{Q}_2 : x_+ \notin \widetilde{\mathbf{Max}}\}$, which is a set of full (Haar) measure.

Denote $\widehat{M} = M_0$ and define $M_i = S^i M_0 S^{-i}$; clearly, we have

$$(12) \quad T_i \circ \widehat{D} = \widehat{D} \circ M_i,$$

because \widehat{D} commutes with S .

Theorem 3.2. *The group of transformations, generated by $M_i, i \in \mathbb{Z}$, is algebraically isomorphic to the group Q_2 :*

$$(13) \quad M_{i+1} = M_i^2, \quad i \in \mathbb{Z}.$$

Thus, we get a new (Morse) action of the group Q_2 on \widehat{Q}_2 . For all i , M_i is a 2-point extension of T_i .

Proof. We only need to check (13); other statements follow immediately. Using $S\widehat{D} = \widehat{D}S$ and $T_{i+1} = T_i^2$, we obtain that $\widehat{D}M_{i+1} = \widehat{D}M_i^2$. It remains to observe that $M_{i+1}(\hat{x})$ and $M_i^2(\hat{x})$ are cofinal (agree sufficiently far to the right) for all $x \notin \widetilde{\text{Max}}$. \square

We have defined two canonical measure-preserving actions of the solvable group $\mathbb{Z} \ltimes Q_2$ on \widehat{Q}_2 — the first is generated by the odometer (this is an algebraic action), and the second which is generated by the Morse action. Remember that the Morse action is continuous only almost everywhere.

Questions.

1. Find the cocycle which defines the Morse action as a 2-extension of the algebraic action analogously to formula (5).
2. Give formula analogous to the formula of Theorem 2.7 which gives the Morse action on the solenoid as a time change of the algebraic action.
3. How can we characterize both actions of the group $\mathbb{Z} \ltimes Q_2$ in an intrinsic way?

REFERENCES

- [1] J.-P. Allouche and J. Shallit, *Automatic sequences. Theory, applications, generalizations*, Cambridge University Press, Cambridge, 2003.
- [2] V.I. Arnold. Complexity of finite sequences of zero and ones and geometry of finite space of functions *Funct. Anal. Other Math.* **1** (2006), no. 1, 1-15.
- [3] R. Belinskaya. Partitions of Lebesgue space on trajectories defined by ergodic automorphisms. *Func. Anal. and its Appl.* **2** 1968, no. 1, 190-199.
- [4] F. Durand, B. Host, and C. Skau, Substitutional dynamical systems, Bratteli diagrams and dimension groups, *Ergodic Theory Dynam. Syst.*, **19** (1999), 953–993.
- [5] A. Forrest, K -groups associated with substitution minimal systems, *Isr. J. Math.*, **98** (1997), 101–139.

- [6] S. Kakutani, Ergodic theory on shift transformations, *Proc. 5th Berkeley Symp. Math. Statist. Probab. II*, Berkeley, CA, 1967, pp. 405–414.
- [7] Y. Moshe, On the subword complexity of Thue-Morse polynomial extractions, *Theoret. Comput. Sci.* **389** (2007), no. 1-2, 318–329.
- [8] N. Pytheas-Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, Lecture Notes in Math. **1794**, Springer-Verlag, 2002.
- [9] M. Queffelec, *Substitution Dynamical Systems - spectral analysis*, Springer Lecture Notes in Mathematics, **1294** (1987) 1–240.
- [10] V. A. Rokhlin, Exact endomorphisms of a Lebesgue space (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **25** 1961, 499–530.
- [11] A. M. Vershik, Uniform algebraic approximations of shift and multiplication operators. *Dokl. Akad. Nauk SSSR* 259, No.3, 526-529 (1981). English translation: *Sov. Math. Dokl.* 24, 97-100 (1981).
- [12] A. M. Vershik, Locally transversal symbolic dynamics, (Russian) *Algebra i Analiz* **6** (1994), no. 3, 94–106; English translation: *St. Petersburg Math. J.* **6** (1995), no. 3, 529–540.
- [13] A. M. Vershik, A.N. Livshits, Adic models of ergodic transformations, spectral theory, and related topics, *Adv. in Soviet Math. AMS Transl.* **9** (1992), 185–204.

ANATOLY VERSHIK, MATHEMATICAL INSTITUTE OF RUSSIAN ACADEMY OF SCIENCES, ST.PETERSBURG
BRANCH, FONTANKA 27, ST.PETERSBURG, 191023, RUSSIA

E-mail address: `vershik@pdmi.ras.ru`

BORIS SOLOMYAK, BOX 354350, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON,
SEATTLE WA 98195

E-mail address: `solomyak@math.washington.edu`